Preconditioned Iterative Solvers for the Generalized Stokes Problem *

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Abstract

Discretization of Navier-Stokes equation for incompressible fluids gives rise to large, sparse systems of the form

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{1}$$

where $u \in R^{n\times n}$ is the fluid velocity, $p \in R^m$ is the pressure, and $A \in R^{n\times n}$ such that

$$A = \frac{1}{\Delta t}M + C + \frac{1}{R}L,$$

in which M is the mass matrix, L is the Laplace matrix, and C is the matrix arising from the convection term. The matrix C is zero for Generalized Stokes problem. This system is indefinite because of the incompressibility constraint. This indefiniteness is the main cause of difficulty in solving the system via preconditioned iterative methods. The degree of difficulty also depends upon the nature of A matrix which is affected by the choice of time step Δt, Reynolds number, and fluid velocity.

A discrete divergence-free or solenoidal basis for fluid velocity has the advantage of automatically satisfying the incompressibility constraint. Such a basis can be obtained by computing the null space of the divergence operator B^T. A matrix $P \in R^{n\times(n-m)}$ that satisfies the condition $B^T P = 0$ is used to compute divergence-free velocity via the matrix-vector product $u = Px$,

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for any arbitrary $x \in R^{n-m}$. Clearly such a velocity satisfies the incompressibility constraint $B^T u = 0$, and hence, the linear system (1) reduce to

$$P^T A P x = P^T f. \tag{2}$$

This system can be solved by suitable iterative methods such as CG or GMRES.

The success of the solenoidal basis method depends on several factors including fast computation of matrix-vector products and effective preconditioning of the matrix $P^T AP$. In this paper, we outline the work already done [2, 3] in computing the matrix P using local solenoidal functions and computing the associated matrix-vector products in a matrix free form [1]. Developing a good preconditioner is a challenging task since the reduced system is not computed explicitly. This paper outlines a preconditioning technique that results in convergence in almost constant number of iterations regardless of various parameters such as the mesh width, Reynolds number, and the time step. Experiments performed on SGI Origin 2000 demonstrate the effectiveness of the proposed scheme and its parallel implementation.

References

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