A class of spectral two-level preconditioners

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When solving the linear system $Ax = b$ with a Krylov method, the smallest eigenvalues of the matrix A often slow down the convergence. In the SPD case, this is clearly highlighted by the bound on the rate of convergence of the Conjugate Gradient method (CG) given by

$$||e^{(k)}||_A \leq \left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1}\right)^k ||e^{(0)}||_A, \tag{1}$$

where $e^{(k)} = x^* - x^{(k)}$ denotes the forward error associated with the iterate at step k and $\kappa(A) = \dfrac{\lambda_{max}}{\lambda_{min}}$ denotes the condition number. From this bound it can be said that enlarging the smallest eigenvalues would improve the convergence rate of CG. Consequently if the smallest eigenvalues of A could be somehow "removed" the convergence of CG will be improved. Similarly for unsymmetric systems arguments exist to explain the bad effect of the smallest eigenvalues on the rate of convergence of the unsymmetric Krylov solver [1, 3, 5]. To cure this, several techniques have been proposed in the last few years, mainly to improve the convergence of GMRES. In [5], it is proposed to add a basis of the invariant space associated with the smallest eigenvalues to the Krylov basis generated by GMRES. Another approach based on a low rank update of the preconditionner for GMRES was proposed by [1, 3]. They consider the orthogonal complement of the invariant subspace associated with the smallest eigenvalues to build a low rank update of the preconditioned system. Finally, in [4] a preconditioner for GMRES based on a sequence of rank-one updates is proposed that involves the left and right smallest eigenvectors. In our work, we consider an explicit eigencomputation which makes the preconditioner independent of the Krylov solver used in the actual solution of the linear system.

We first present our techniques for unsymmetric linear systems and then derive a variant for symmetric and SPD matrices. We consider the solution of the linear system

$$Ax = b, \tag{2}$$

where A is a $n \times n$ unsymmetric non singular matrix, x and b are vectors of size n. The linear system is solved using a preconditioned Krylov solver and we denote by M_1 the left preconditioner, meaning that we solve

$$M_1 Ax = M_1 b. \tag{3}$$

We assume that the preconditioned matrix $M_1 A$ is diagonalizable, that is:

$$M_1 A = V \Lambda V^{-1}, \tag{4}$$

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with $\Lambda = diag(\lambda_i)$, where $|\lambda_1| \leq \ldots \leq |\lambda_n|$ are the eigenvalues and $V = (v_i)$ the associated right eigenvectors. We denote by $U = (u_i)$ the associated left eigenvectors; we then have $U^H V = diag(u_i^H v_i)$, with $u_i^H v_i \neq 0$, $\forall i$ [6]. Let V_ε be the set of right eigenvectors associated with the set of eigenvalues λ_i such that $|\lambda_i| \leq \varepsilon$. Similarly we define by U_ε the corresponding subset of left eigenvectors. We define by $A_c = U_\varepsilon^H M_1 A V_\varepsilon$, $M_c = V_\varepsilon A_c^{-1} U_\varepsilon^H M_1$ and finally $M = M_1 + M_c$. It can be shown that MA is diagonalisable and we have $MA = V^{-1} diag(\eta_i) V$ with

$$\begin{cases} \eta_i = \lambda_i & \text{if} \quad |\lambda_i| > \varepsilon, \\ \eta_i = 1 + \lambda_i & \text{if} \quad |\lambda_i| \leq \varepsilon. \end{cases} \tag{5}$$

If only the eigenvalues have to be shifted without preserving the associated invariant subspaces, the following result can be shown: Let W be such that $\tilde{A}_c = W^H A V_\varepsilon$ has full rank, $\tilde{M}_c = V_\varepsilon \tilde{A}_c^{-1} W^H$ and finally $\tilde{M} = M_1 + \tilde{M}_c$. Then $\tilde{M}A$ is similar to a matrix whose eigenvalues are the ones given in (5).

For right preconditioning, that is $AM_1 y = b$, similar results can be established. We should point out that if the symmetry of the preconditioner has to be preserved an obvious choice exists. For left preconditioning we can set $W = V_\varepsilon$, that nevertheless does not imply that A_c has full rank. For SPD matrices this choice leads to a SPD preconditioner. Indeed the preconditioner $\tilde{M}$ is the sum of a SPD matrix M_1 and the low rank update that is symmetric semi-definite; we point out that in this case the preconditioner has a similar form to the one proposed in [2] for two-level preconditioners in non-overlapping domain decomposition.

In this talk we will give a sketch for the proofs of the results introduced above and show the numerical efficiency of the proposed two-level preconditioner on:

1. sparse unsymmetric matrices from the RB matrix collection or arising from the discretization of model convection diffusion equations,

2. sparse block structured unsymmetric and SPD matrices arising from non-overlapping domain decomposition techniques in semiconductor device modelling,

3. dense symmetric complex non-Hermitian matrices arising from electromagnetism calculations.

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