Least-Squares Mixed Finite Element Solution of Unsaturated-Saturated Subsurface Flow Problems

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Abstract

A least-squares mixed finite element formulation is applied to the nonlinear elliptic problems arising in each time-step of an implicit Euler discretization for unsaturated-saturated flow problems. This approach simultaneously constructs approximations to the flux in Raviart-Thomas spaces and to the hydraulic potential by standard H^1-conforming linear finite elements. Two important properties of the least-squares approach are investigated in detail: the local least-squares functional provides an a posteriori error estimator, and Gauß-Newton methods are robust iterative solvers for the resulting nonlinear least-squares problems. Computational experiments conducted for a realistic water table recharge problem illustrate the effectiveness of this approach.

1 Introduction

This paper contains a study of least-squares mixed finite element methods applied to the highly nonlinear elliptic boundary value problems which arise in the course of solving unsaturated-saturated subsurface flow problems. We restrict ourselves to two-dimensional problems associated with a vertical cross-section of the subsurface and concentrate on the discretization in space. The physical quantities which describe this system are the hydraulic potential and the volumetric flux, the latter of which is often of special interest. Therefore it is natural to use a mixed finite element approach which computes approximations in finite-dimensional spaces to both the pressure and the flux (see [5]).

In this paper, the least-squares finite element methodology for first-order systems (see e.g. [3]) is applied to a nonlinear problem arising from modelling unsaturated-saturated subsurface flow. Two important properties of the least-squares approach which will be investigated in detail are: (i) the use of the local evaluation of the (nonlinear) least-squares functional as an a posteriori error estimator, and (ii) the robustness of the iterative solution of the resulting nonlinear least-squares problems by a Gauß-Newton method. Adaptive refinement strategies based on the local least-squares functional allow the resolution of the steep saturation fronts which occur as water infiltrates dry soil.

The following section presents the necessary background on the least-squares formulation of the system arising in unsaturated-saturated subsurface flow including a proof of the finite element approximation properties. Section 3 discusses the suitability of the local least-squares functional as an a posteriori error estimator and the resulting refinement strategy. In Section 4 the iterative solution of the resulting nonlinear least-squares problems by a Gauß-Newton method is described. Finally, Section 5 presents the results of computational experiments conducted for a realistic water table recharge problem which resembles one of the standard test problems for unsaturated-saturated subsurface flow from [12].

2 Nonlinear Least-Squares Formulation

We consider unsaturated-saturated flow in soil modelled by the first-order system

$$\begin{aligned} \frac{\partial \theta(p)}{\partial t} + \operatorname{div}\, u + s &= 0 \\ u + K(p)\, \nabla p &= 0 \end{aligned} \tag{1}$$

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for the hydraulic potential $p : \Omega \to \mathbb{R}$ and the volumetric flux $u : \Omega \to \mathbb{R}^2$ in a domain $\Omega \subset \mathbb{R}^2$ subject to appropriate initial and boundary conditions. Here $\theta(p)$ denotes the water content, $K(p)$ the permeability (both of which are bounded) and s a source or sink term. Of course, all of these functions also depend on the point $x \in \Omega$ which is not explicitly stated here for notational simplicity. For instance, K and θ depend nonlinearly on p using the model by Mualem [9] and van Genuchten [11] and may be viewed as piecewise constant in areas of different soil types. The first equation in (1) stands for conservation of mass, the second one which links flux to pressure is the Buckingham-Darcy law (see, for example, [1]). On the boundary of Ω either the pressure or the normal flux is prescribed: $\partial\Omega = \Gamma_N \cup \Gamma_D$ with $n \cdot u = g$ on Γ_N and $p = h$ on Γ_D.

Discretizing in time by an implicit Euler scheme leads to the system

$$
\begin{aligned}
\frac{\theta(p) - \theta(p^{\mathrm{old}})}{\Delta t} + \mathrm{div}\ u + s &= 0 \\
u + K(p)\nabla p &= 0
\end{aligned}
\tag{2}
$$

for u and p at the new time-step $t = t^{\mathrm{old}} + \Delta t$. This is a nonlinear elliptic boundary value problem written as a first-order system. Under the assumption that $s \in L^2(\Omega)$, we may define, for $u \in H(\mathrm{div}\ ,\Omega)$ and $p \in H^1(\Omega)$, the least-squares functional

$$
\mathcal{F}(u, p; p^{\mathrm{old}}, s) = \|\frac{\theta(p) - \theta(p^{\mathrm{old}})}{\Delta t} + \mathrm{div}\ u + s\|^2_{0,\Omega} + \|\sqrt{w}(u + K(p)\ \nabla p)\|^2_{0,\Omega}\ . \tag{3}
$$

Obviously, each solution of (2) is a minimum of the least-squares functional in (3). The weight function w is assumed to be bounded from below in Ω by a positive constant. In order to account for the large jumps of $K(p)$ between subregions consisting of different soils, it is a good idea to choose $w = 1/K_s$ where K_s is the permeability in the saturated case. For the steady-state linear (fully saturated) problem, this is motivated by ellipticity considerations of the corresponding least-squares functional

$$
\bar{\mathcal{F}}(u, p; s) = \|\mathrm{div}\ u + s\|^2_{0,\Omega} + \|\sqrt{w}(u + K(p)\ \nabla p)\|^2_{0,\Omega}\ .
$$

in [8].

After choosing appropriate finite-dimensional spaces

$$
\begin{aligned}
V_h &\subset \{v \in H(\mathrm{div}\ ,\Omega) : n \cdot v = 0 \text{ on } \Gamma_N\}\ , \\
Q_h &\subset \{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_D\}\ ,
\end{aligned}
$$

we compute approximate solutions $u_h = u_h^N + \hat{u}_h$ and $p_h = p_h^D + \hat{p}_h$, where u_h^N and p_h^D satisfy the Neumann and Dirichlet boundary conditions, respectively, and with $\hat{u}_h \in V_h$ and $\hat{p}_h \in Q_h$. The least-squares approach suggests to choose $\hat{u}_h$ and $\hat{p}_h$ such that $\mathcal{F}(u_h, p_h; p_h^{\mathrm{old}}, s)$ is minimized among all u_h and p_h of the above form. The simplest choice is to use the lowest-order Raviart-Thomas space for V_h and standard continuous piecewise functions for Q_h on a triangulation $\mathcal{T}_h$ of Ω. Note that the spaces Q_h for the pressure unknowns are chosen in $H^1(\Omega)$ (rather than in $L^2(\Omega)$ for the standard mixed method) which makes sense from a physical point of view and allows the explicit enforcement of the Dirichlet boundary conditions. There are more aspects in which this approach differs from the standard mixed finite element method (see [5]), as we shall see in the sequel. Note that it is also possible to impose the boundary conditions only weakly by appropriate least-squares boundary functionals as described in [10].

Under the assumption that the system (2) has a unique solution, we have

$$
\mathcal{F}(u_h, p_h; p^{\mathrm{alt}}, s) = \|\frac{\theta(p_h) - \theta(p)}{\Delta t} + \mathrm{div}\ (u_h - u)\|^2_{0,\Omega} + \|\sqrt{w}(u_h - u + K(p_h)\ \nabla p_h - K(p)\ \nabla p)\|^2_{0,\Omega}\ .
$$

Linearization leads to

$$
\begin{aligned}
&\mathcal{F}(u_h, p_h; p^{\mathrm{alt}}, s) \\
\approx &\|\frac{\theta'(p)}{\Delta t}\ (p_h - p) + \mathrm{div}\ (u_h - u)\|^2_{0,\Omega} + \|\sqrt{w}(u_h - u + K'(p)\ \nabla p\ (p_h - p) + K(p)\ \nabla(p_h - p))\|^2_{0,\Omega} \\
= &\|\frac{\theta'(p)}{\Delta t}\ (p_h - p) + \mathrm{div}\ (u_h - u)\|^2_{0,\Omega} + \|\sqrt{w}(u_h - u + \nabla(K(p)\ \nabla(p_h - p)))\|^2_{0,\Omega}
\end{aligned}
$$

if p_h and u_h are sufficiently close to p and u. For the model that we use in our computations (based on [9, 11]) we have $\|\theta'(p)\|_{L^\infty(\Omega)}$ and $\|K'(p)\|_{L^\infty(\Omega)}$ bounded independently of p. In dry regions $K(p)$ as well as $K'(p)$ and $\theta'(p)$ may be very small. In this case, the first-order system (2) effectively reduces to div $u+s=0$ and $u=0$ leaving p completely undetermined. We regularize the problem using $K_{\min}+(1-K_{\min}/K_s)K(p)$ with $K_{\min}=10^{-3}K_s$ instead of $K(p)$ for the permeability which implies that $0<K_{\min}\le K(p)\le K_s$ holds uniformly in Ω. For simplicity we assume that w is constant throughout Ω, i.e., only one type of soil is present. The results below can be extended to variable w, i.e., jumps in the permeabilities, using the framework in [8] but this is not our main issue here. Moreover, we assume that $\Gamma_D\subset\partial\Omega$ is the union of arcs of positive length such that a Poincaré-Friedrichs inequality

$$\|K(p)\,q\|_{0,\Omega}\le C_F\|\nabla(K(p)\,q)\|_{0,\Omega} \tag{4}$$

with a constant $C_F>0$ holds for all $q\in H^1_{\Gamma_D}(\Omega)$. Under these assumptions we may prove that the least-squares functional is equivalent to the natural norms of the error $(u-u_h,p-p_h)\in H(\mathrm{div},\Omega)\times H^1(\Omega)$. We use the notation $\xi_h\lesssim\eta_h$ to indicate that $\xi_h\le c\eta_h$ holds with a constant c which is independent of h.

Theorem 1 *Under the above assumptions,*

$$\begin{aligned}
&\|u-u_h\|^2_{\mathrm{div},\Omega}+\|K(p)(p-p_h)\|^2_{1,\Omega}-\|\frac{\theta'(p)}{\Delta t}(p-p_h)\|^2_{0,\Omega}\\
\lesssim\ &\|\frac{\theta'(p)}{\Delta t}\,(p_h-p)+\mathrm{div}\ (u_h-u)\|^2_{0,\Omega}+\|\sqrt{w}(u_h-u+\nabla(K(p)\,(p_h-p)))\|^2_{0,\Omega}\\
\lesssim\ &\|u-u_h\|^2_{\mathrm{div},\Omega}+\|K(p)(p-p_h)\|^2_{1,\Omega}+\|\frac{\theta'(p)}{\Delta t}(p-p_h)\|^2_{0,\Omega}\,.
\end{aligned}$$

Proof. Assume for simplicity that $w\equiv 1$. Using

$$(u_h-u,\nabla(K(p)\,(p_h-p)))_{0,\Omega}+(\mathrm{div}\ (u_h-u),K(p)\,(p_h-p))_{0,\Omega}=0$$

(note that $(n\cdot(u_h-u))(p_h-p)=0$ on $\partial\Omega$), (4) and elementary algebraic manipulations lead to

$$\begin{aligned}
&\|\frac{\theta'(p)}{\Delta t}\,(p_h-p)+\mathrm{div}\ (u_h-u)\|^2_{0,\Omega}+\|u_h-u+\nabla(K(p)\,(p_h-p))\|^2_{0,\Omega}\\
&=\left(1-\varepsilon_1-\frac{\delta}{\varepsilon_2}\right)\ \|\mathrm{div}\ (u_h-u)\|^2_{0,\Omega}+\delta\,\|u_h-u\|^2_{0,\Omega}\\
&+\delta\,\|\nabla(K(p)\,(p_h-p))\|^2_{0,\Omega}-\delta\varepsilon_2\,\|K(p)\,(p_h-p)\|^2_{0,\Omega}-\left(\frac{1}{\varepsilon_1}-1\right)\ \|\frac{\theta'(p)}{\Delta t}\,(p_h-p)\|^2_{0,\Omega}\\
&+\|\sqrt{\varepsilon_1}\ \mathrm{div}\ (u_h-u)+\frac{1}{\sqrt{\varepsilon_1}}\frac{\theta'(p)}{\Delta t}\,(p_h-p)\|^2_{0,\Omega}\\
&+(1-\delta)\,\|u_h-u+\nabla(K(p)\,(p_h-p)))\|^2_{0,\Omega}+\delta\,\|\frac{1}{\sqrt{\varepsilon_2}}\ \mathrm{div}\ (u_h-u)-\sqrt{\varepsilon_2}\,K(p)\,(p_h-p)\|^2_{0,\Omega}
\end{aligned}$$

$$\begin{aligned}
&\ge\left(1-\varepsilon_1-\frac{\delta}{\varepsilon_2}\right)\ \|\mathrm{div}\ (u_h-u)\|^2_{0,\Omega}+\delta\,\|u_h-u\|^2_{0,\Omega}\\
&+\delta(1-\varepsilon_2C_F^2)\,\|\nabla(K(p)\,(p_h-p))\|^2_{0,\Omega}-\left(\frac{1}{\varepsilon_1}-1\right)\ \|\frac{\theta'(p)}{\Delta t}\,(p_h-p)\|^2_{0,\Omega}
\end{aligned}$$

with $\delta,\varepsilon_1,\varepsilon_2\in(0,1)$. Choosing

$$\delta=\min\{\frac{1}{8},\frac{1}{8C_F^2}\}\ ,\ \varepsilon_1=\frac{1}{2}\ ,\ \varepsilon_2=\min\{\frac{1}{2},\frac{1}{2C_F^2}\}$$

leads to the first inequality in Theorem 1. The second inequality follows from

$$\begin{aligned}
&\|\frac{1}{\Delta t}\theta'(p)\,(p_h-p)+\mathrm{div}\ (u_h-u)\|^2_{0,\Omega}+\|u_h-u+\nabla(K(p)\,(p_h-p))\|^2_{0,\Omega}\\
&\le 2\|\frac{1}{\Delta t}\theta'(p)\,(p_h-p)\|^2_{0,\Omega}+2\|\mathrm{div}\ (u_h-u)\|^2_{0,\Omega}+2\|u_h-u\|^2_{0,\Omega}+2\|\nabla(K(p)\,(p_h-p))\|^2_{0,\Omega}\,.
\end{aligned}$$

□

For the parametrization by Mualem and van Genuchten [9, 11], $\theta'(p)/K(p)$ is bounded and therefore,

$$\|\frac{1}{\Delta t}\theta'(p)\,(p_h - p)\|_{0,\Omega}^2 \lesssim \|K(p)\,(p_h - p)\|_{0,\Omega}^2 \,.$$

For the solution of elliptic boundary value problems of the type (2) we have $u \in H^\alpha(\Omega), p \in H^{1+\alpha}(\Omega)$ for some $\alpha > 0$ which depends on the coefficients $K(p), \theta(p)$ and s and on the geometry of the region. Combined with Theorem 1 and Céa's lemma (in the general form of [2, Theorem 5.7.6]) imply that, for sufficiently small h,

$$\begin{aligned} |||(u - u_h, p - p_h)||| &:= \left(\|u - u_h\|_{\mathrm{div},\Omega}^2 + \|K(p)(p - p_h)\|_{1,\Omega}^2\right)^{1/2} \\ &\lesssim h^\alpha \left(\|u\|_{\alpha,\Omega}^2 + \|\mathrm{div}\ u\|_{\alpha,\Omega}^2 + \|p\|_{1+\alpha,\Omega}^2\right)^{1/2} \,. \end{aligned} \tag{5}$$

3 Adaptive Mesh Refinement

Clearly, the least-squares functional is the sum of its contributions from each triangle $T \in \mathcal{T}_h$,

$$\mathcal{F}(u_h, p_h; p^{\mathrm{alt}}, s) = \sum_{T \in \mathcal{T}_h} \mathcal{F}_T(u_h, p_h; p^{\mathrm{alt}}, s)$$

with

$$\mathcal{F}_T(p_h, u_h; p^{\mathrm{alt}}, s) = \|\frac{1}{\Delta t}(\theta(p_h) - \theta(p^{\mathrm{alt}})) + \mathrm{div}\ u_h + s\|_{0,T}^2 + \|\sqrt{w}(u_h + K(p_h)\,\nabla p_h)\|_{0,T}^2 \,.$$

Combined with the fact that, with a norm $|||(\cdot,\cdot)|||$ that we accept as an appropriate error measure on $H(\mathrm{div}, \Omega) \times H^1(\Omega)$,

$$\sum_{T \in \mathcal{T}_h} \mathcal{F}_T(u_h, p_h; p^{\mathrm{alt}}, s) \sim |||(u - u_h, p - p_h)|||^2$$

this means that the local evaluation of the least-squares functional constitutes an a posteriori error estimator (cf. [13, Chap. 1]). In fact, using the least-squares functional as an error estimator has several advantages compared to other techniques:

(i) The local evaluation of the least-squares functional is readily available and no additional computations have to performed.

(ii) Evaluation of the nonlinear least-squares functional means that the error with respect to the nonlinear problem is estimated directly and no linearization of the problem has to be done.

(iii) There is only one step where (unknown) constants are involved. After accepting the least-squares functional as an appropriate error measure, a prescribed accuracy can be achieved by an adaptive refinement strategy based on computing the error measure (the least-squares functional) after each refinement step.

If our goal is to achieve $\mathcal{F}(u_h, p_h; p^{\mathrm{alt}}, s) \le \varepsilon$, a natural refinement strategy is to refine those triangles with $\mathcal{F}_T(u_h, p_h; p^{\mathrm{alt}}, s) > \varepsilon/(2(\#T))$ where $\#T$ is the number of triangles in the current triangulation.

4 Gauß-Newton Method

The Gauß-Newton approach consists of replacing the nonlinear least-squares problem by a sequence of linear ones as follows: Our aim is to replace the current iterate $(u_h^{(k)}, p_h^{(k)})$ by an update $(u_h^{(k+1)}, p_h^{(k+1)}) =$

$(u_h^{(k)} + \Delta u_h, p_h^{(k)} + \Delta p_h)$ with $\Delta u_h \in V_h, \Delta p_h \in Q_h$ such that the least-squares functional is reduced. Linear approximation leads to

$$\mathcal{F}(u_h^{(k)} + \Delta u_h, p_h^{(k)} + \Delta p_h; p_h^{\text{old}}, s) \approx \|\text{div } u_h^{(k)} + \text{div } \Delta u_h + \frac{\theta(p_h^{(k)}) - \theta(p_h^{\text{old}})}{\Delta t} + \frac{\theta'(p_h^{(k)})}{\Delta t}\Delta p_h + s\|_{0,\Omega}^2$$
$$+ \|\sqrt{w}(u_h^{(k)} + \Delta u_h + K(p_h^{(k)})\, \nabla p_h^{(k)} + K'(p_h^{(k)})\, \nabla p_h^{(k)}\, \Delta p_h + K(p_h^{(k)})\, \nabla \Delta p_h)\|_{0,\Omega}^2 \, .$$

Minimizing this linear least-squares functional is equivalent to finding $\Delta u_h \in V_h$ and $\Delta p_h \in Q_h$ such that

$$\begin{aligned}
0 &= (\text{div } \Delta u_h + \text{div } u_h^{(k)} + \frac{\theta'(p_h^{(k)})}{\Delta t}\Delta p_h + \frac{\theta(p_h^{(k)}) - \theta(p_h^{\text{old}})}{\Delta t} + s, \text{div } v_h)_{0,\Omega} \\
&\quad + (w(\Delta u_h + u_h^{(k)} + K'(p_h^{(k)})\, \nabla p_h^{(k)}\, \Delta p_h + K(p_h^{(k)})\, \nabla \Delta p_h + K(p_h^{(k)})\, \nabla p_h^{(k)}), v_h)_{0,\Omega} \\
&\quad \text{for all } v_h \in \hat{V}_h \, , \\
0 &= (\frac{\theta'(p_h^{(k)})}{\Delta t}(\text{div } \Delta u_h + \text{div } u_h^{(k)} + \frac{\theta'(p_h^{(k)})}{\Delta t}\Delta p_h + \frac{\theta(p_h^{(k)}) - \theta(p_h^{\text{old}})}{\Delta t} + s), q_h)_{0,\Omega} \\
&\quad + (w(\Delta u_h + u_h^{(k)} + K'(p_h^{(k)})\nabla p_h^{(k)} \Delta p_h + K(p_h^{(k)})\nabla \Delta p_h + K(p_h^{(k)})\nabla p_h^{(k)}), \\
&\qquad\qquad K'(p_h^{(k)})\nabla p_h^{(k)} q_h + K(p_h^{(k)})\nabla q_h)_{0,\Omega} \\
&\quad \text{for all } q_h \in \hat{W}_h \, .
\end{aligned} \tag{6}$$

As a stopping criterion for the Gauß-Newton iteration we use

$$1 - \frac{\mathcal{F}(u_h^{(k)}, p_h^{(k)})}{\mathcal{F}(u_h^{(k-1)}, p_h^{(k-1)})} \leq \varepsilon \tag{7}$$

with some prescribed tolerance ε. Note that we cannot use the functional $\mathcal{F}(u_h^{(k)}, p_h^{(k)})$ itself as a stopping criterion since its minimum with respect to discrete spaces is not zero. Alternatively, the residual with respect to the variational problem (6) could be used as a stopping criterion. However, we view the change of the least-squares functional as a more natural measure of the improvement of successive iterates.

Another important issue is the proper choice of damping parameters in order to get global convergence of the Gauß-Newton method. We use the following variant of the Goldstein-Armijo line search principle (cf. e.g. [7, Sect. 4.3.2]): Successively choose μ_k from $\{1, 1/2, 1/4, \ldots, \mu_{\min}\}$ and set

$$(u_h^{(k+1)}, p_h^{(k+1)}) = (u_h^{(k)} + \mu_k \Delta u_h, p_h^{(k)} + \mu_k \Delta p_h)$$

until

$$0 \leq \mathcal{F}(u_h^{(k)}, p_h^{(k)}) - \mathcal{F}(u_h^{(k+1)}, p_h^{(k+1)}) \leq \left(1 - \frac{\mu_k}{2}\right) (\mathcal{F}(u_h^{(k-1)}, p_h^{(k-1)}) - \mathcal{F}(u_h^{(k+1)}, p_h^{(k+1)})) \tag{8}$$

is satisfied. Note that the cost of the evaluation of the least-squares functional $F(u_h^{(k+1)}, p_h^{(k+1)})$ is negligible compared to the computation of the search direction $(\Delta u_h, \Delta p_h)$.

5 Computational Experiments

For our numerical tests, we use the setting in [12] of a water table recharge problem. The domain Ω is a box of 3 m length and 2 m depth which is filled with sand. At time $t = 0$ we assume a constant hydraulic potential at -1.7 m (which is the position of the groundwater table). For $t > 0$ we have constant infiltration of 0.148 m/h through the boundary segment Γ_1. Zero flux boundary conditions are prescribed at the boundary segments Γ_2, Γ_3 and Γ_4, while the hydraulic potential is held constant at -1.7 m at the right boundary Γ_5. Note that these boundary conditions are different from the ones in [12] where a lateral trench (with rising water level) was modelled on the right boundary. Moreover, the value for the saturated permeability used in [12] differs somewhat from the parametrization described below.

We use the model by van Genuchten [11] with the parameters listed in [4] for the functions $K(p)$ and $\theta(p)$. For some types of soil (e.g. sand), this parametrization leads to unbounded $K'(p)$ in the saturated limit which causes problems for the Newton iteration. Following a suggestion in [6, Chap. 6], we replace $K(p)$ by a cubic spline interpolant in a neighborhood of the saturated limit (for $0 \le z - p \le 10^{-2}$ where z is the height). We also found it useful to limit the permeability from below by $10^{-3} K_s$ as indicated in Section 2.

Figure 1 shows the results of our simulation using the least-squares approach. The time-step is usually chosen adaptively using a strategy which is based on the change of water content and pressure head. For this test problem, such a strategy indicated that a constant time-step of $\Delta t = 0.05$ hours (which we use for simplicity) is not too bad of a choice. The graphs on the left show the saturation, on the right we have the triangulations after three steps of adaptive refinement based on the element-wise evaluation of the least-squares functional. The refinement strategy used is as follows: For each time-step we start at the same coarse triangulation and successively minimize the nonlinear least-squares functional. The triangulations are constructed according to the refinement rule of Section 3 with $\varepsilon = 5 \cdot 10^{-4}$.

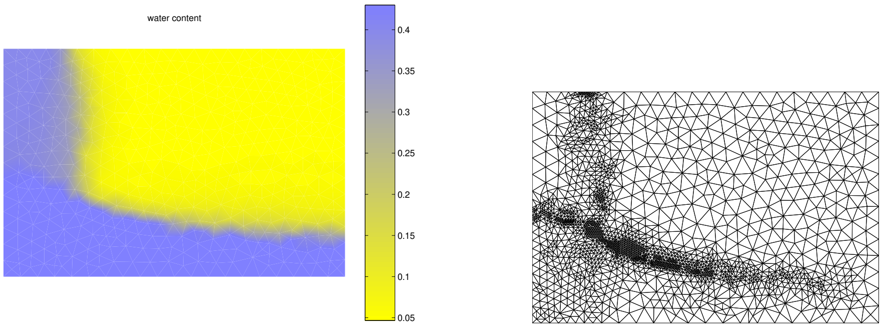

Figure 1: Saturation profile (left) and triangulation after 3 steps of adaptive refinements (right) after 8 hours

Table 1 shows the numbers obtained on the triangulations on level 0 to 3. It is apparent that the least-squares functional which is our error measure is reduced quite effectively by adding new degrees of freedom where suggested from our refinement strategy. The last table shows the number of Gauß-Newton iterations required to fulfill (7) with $\varepsilon = 10^{-4}$. On each level, the iteration is started with $(u_h^{\rm old}, p_h^{\rm old})$, i.e., the approximation (on the coarsest triangulation) from the previous time step. This is to illustrate the dependence of the Gauß-Newton steps on the size of the triangulation. In order to save computational time, of course, one should start the iteration on level l, for $l \ge 1$, using as a starting guess the approximation from level $l-1$. Nevertheless, the number of Gauß-Newton iterations seems to be bounded independently of the refinement level. Moreover, less than 10 Gauß-Newton iterations are required in most cases and at most 18 in all cases except on the coarsest level at the beginning of the simulation. The rather large numbers 48 and 38 after 1 and 2 hours, respectively, for $l = 0$ may be caused by a rather poor linearization on such a coarse triangulation.

We think that these numerical results show the effectiveness of the least-squares mixed finite element approach to these types of nonlinear problems. In particular, the numbers listed in Table 1 clearly demonstrate two favorable properties: (i) the use of the (local) least-squares functional as an error estimator leads to an efficient adaptive mesh refinement technique, and (ii) Gauß-Newton methods are robust iterative schemes to solve the resulting nonlinear least-squares problems.

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Table 1: Degrees of freedoms ($\dim V_h + \dim Q_h$) vs. minimum of least-squares functional

t	2 hours	4 hours	6 hours	8 hours	10 hours
$l=0$	705 + 267	705 + 267	705 + 267	705 + 267	705 + 267
$l=1$	1084 + 402	1245 + 455	1419 + 515	1829 + 657	1861 + 665
$l=2$	1890 + 680	2288 + 812	3013 + 1057	4588 + 1592	4782 + 1658
$l=3$	3594 + 1260	4615 + 1599	6227 + 2141	10464 + 3574	10189 + 3473
$l=4$	7178 + 2466	9059 + 3089	11823 + 4017	21009 + 7105	20803 + 7033

t	2 hours	4 hours	6 hours	8 hours	10 hours
$l=0$	$4.25 \cdot 10^{-3}$	$2.81 \cdot 10^{-3}$	$2.36 \cdot 10^{-3}$	$2.55 \cdot 10^{-3}$	$2.19 \cdot 10^{-3}$
$l=1$	$2.59 \cdot 10^{-3}$	$1.52 \cdot 10^{-3}$	$1.14 \cdot 10^{-3}$	$1.14 \cdot 10^{-3}$	$1.03 \cdot 10^{-3}$
$l=2$	$1.62 \cdot 10^{-3}$	$8.59 \cdot 10^{-4}$	$5.89 \cdot 10^{-4}$	$5.95 \cdot 10^{-4}$	$5.18 \cdot 10^{-4}$
$l=3$	$8.88 \cdot 10^{-4}$	$4.69 \cdot 10^{-4}$	$2.86 \cdot 10^{-4}$	$3.10 \cdot 10^{-4}$	$2.64 \cdot 10^{-4}$
$l=4$	$4.22 \cdot 10^{-4}$	$2.41 \cdot 10^{-4}$	$1.46 \cdot 10^{-4}$	$1.56 \cdot 10^{-4}$	$1.31 \cdot 10^{-4}$

t	1 h	2 h	3 h	4 h	5 h	6 h	7 h	8 h	9 h	10 h
$l=0$	48	38	18	12	9	5	4	5	4	5
$l=1$	8	10	9	6	5	5	6	6	6	6
$l=2$	4	5	5	5	5	14	8	8	8	10
$l=3$	7	6	6	6	6	18	12	11	13	12
$l=4$	12	11	11	11	10	16	14	13	15	13

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