On the Convergence of Iterative Methods for Linear Systems arising from Singularly Perturbed Equations

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Abstract

Singularly perturbed differential equations are model equations for convection dominated flow problems. Many numerical methods for such differential equations, particularly those which seek convergence uniformly in the perturbation parameter, utilize some form of fitting or upwinding of the operator. In such cases, the matrices which arise may be highly non-symmetric. Previous studies have shown that, when using Gauss-Seidel type iterations, it is crucial to sweep the mesh in the direction of the underlying flow. In this paper, we will examine iterative methods for singularly perturbed equations further, particularly from the perspective of analysing the dependency of the convergence rate on the number of mesh nodes, and on the perturbation parameter.

1. Introduction

Convection diffusion problems are a class of problems which model fluid flows. The numerical solution of such problems exhibit significant difficulties, particularly when the diffusion coefficient is small, which corresponds to the case of high Reynolds number. Such problems are said to have *convection dominated flow.* The numerical difficulties arise due to the presence of sharp boundary and/or interior layers, which degrade the accuracy of standard difference schemes. One approach to solving these problems is to employ a uniform mesh and either an upwind scheme or an *(exponentially) fitted* scheme which attempts to model the boundary layer or interior layer accurately [1][2][3]. The advantages of these methods arise as a result of the care taken to ensure that the scheme corresponds to the physical process by respecting the natural direction of flow in the problem, as given by the characteristics of the differential equation. If the difference scheme does not conform to this inherent direction in the problem, a satisfactory solution will not be obtained without considerable

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computational effort. An obvious question which arises is whether this directionality in the solution need also be conformed to by an iterative method used to solve the linear systems, arising from the difference scheme.

In this paper, we consider the linear systems which arise as a result of the finite difference discretization, on a uniform mesh, of linear singularly perturbed differential equations in one dimension. Such problems are simple theoretical model problems for the physical phenomena of convection dominated flow. We shall attempt to characterize the behaviour of variants of the Gauss-Seidel iterative method, for various degree of convection dominated flow and in particular we shall present results for the error for each iteration as a function of the perturbation parameter ϵ and the number of mesh intervals n.

2. Model Problem

In this paper we will confine our analysis to a sample one dimensional problem :

$$\begin{aligned} \epsilon u''(x) + p(x)u'(x) - r(x)u(x) = f(x),\ \ 0 < x < 1, \\ \hat{p} \geq p(x) \geq \check{p} > 0, \quad r(x) \geq 0 \\ u(0) = A, \quad u(1) = B \end{aligned} \tag{1}$$

In this case, ϵ plays the role of the diffusion coefficient and we are interested primarily in the case where the flow is convection dominated, that is $\epsilon << 1$, whereas $p(x)$ is of order 1 over most of the region under consideration. For this simple sample problem the flow is to the left and there is an exponential boundary layer at the left hand boundary. We reflect this fact by discretizing using an upwinded scheme in the forward direction or an exponentially fitted scheme. On a uniform grid of size $h = 1/(n+1)$ the upwinded difference scheme, written in the flow direction independent form, is :

$$\begin{gathered} b_i u_{i-1} - a_i u_i + c_i u_{i+1} = f_i,\ \ 1 \leq i \leq n-1, \\ u_0 = A, \quad u_n = B, \end{gathered} \tag{2}$$

where the coefficients a_i, b_i, c_i are given by

$$\begin{aligned} b_i &= \frac{\epsilon}{h^2} + \frac{1}{2h}(|p_i| - p_i) \geq 0, \\ a_i &= \frac{2\epsilon}{h^2} + \frac{1}{h}|p_i| + r_i \geq 0, \\ c_i &= \frac{\epsilon}{h^2} + \frac{1}{2h}(|p_i| + p_i) \geq 0, \end{aligned}$$

and $p_i = p(x_i), r_i = r(x_i)$ and $f_i = f(x_i)$. An exponentially fitted scheme would also be of the form (2), with the coefficients a_i, b_i, c_i given by

$$\begin{aligned} b_i &= \frac{\epsilon_i}{h^2} + \frac{1}{2h}(|p_i| - p_i) \geq 0, \\ a_i &= \frac{2\epsilon_i}{h^2} + \frac{1}{h}|p_i| + r_i \geq 0, \\ c_i &= \frac{\epsilon_i}{h^2} + \frac{1}{2h}(|p_i| + p_i) \geq 0, \end{aligned}$$

where, for example for the Il'in-Allen-Southwell scheme,

$$\epsilon_i = \epsilon\sigma(\frac{|p_i|h}{\epsilon}), \quad \text{and} \quad \sigma(z) = \frac{z}{e^z - 1}.$$

The analysis and results are similar in both cases. However, it is clear that $\epsilon_i < \epsilon$. Thus the exponentially fitted method behaves like an upwinded scheme with smaller ϵ. This makes the difference scheme nodally more accurate than the upwinded scheme. In physical terms, we can view the use of an upwinded scheme, rather than the (unstable) centered difference scheme, as introducing *artificial viscosity* into the approximation. This normally leads to diffusion of the boundary layers. Exponentially fitted schemes effectively seek to use precisely the correct amount of artificial viscosity. For simplicity, we shall do our analysis in terms of the upwinded scheme. However it is clear that analogous results exist for exponentially fitted schemes.

Theoretical Analysis

In either case the matrix $A \in R^{(n-1)\times(n-1)}$, of the difference scheme is a tridiagonal matrix. It is easily shown that this matrix is an *irreducibly diagonally dominant M-matrix* [8, p23] under the above assumptions, and thus that $A^{-1} \geq 0$, where the inequality is meant component-wise.

Now consider the matrix splitting of A, into a diagonal D, a lower triangular L, and an upper triangular U matrix, $A = D - L - U$. The forward and backward point Gauss-Seidel methods, which we shall call FGS and BGS respectively, may now be written as

$$M_f u^{k+1} = N_f u^k + f, \text{ where } \ M_f = D - L, \ N_f = U,$$

$$M_b u^{k+1} = N_b u^k + f, \text{ where } \ M_b = D - U, \ N_b = L.$$

In either case, the splitting $A = M - N$ is a *regular splitting* of the matrix A and $\rho(M^{-1}N) < 1$ [8, Theorem 3.14]. Thus both Gauss-Seidel methods are convergent with the same *asymptotic convergence rate*, since $\rho(M_f^{-1}N_f) = \rho(M_b^{-1}N_b) = \rho(D^{-1}(L+U))^{1/2}$. However although the asymptotic convergence rate is a good measure of the ultimate rate of convergence of the Gauss-Seidel methods, Han et al. [5] [6] and Farrell [4] have shown that the initial rate of convergence for the two methods differ considerably. Thus, the spectral radius, which measures the asymptotic rate of convergence, is not the most suitable measure of the initial behaviour of an iterative scheme, since it is essentially direction independent. For this reason, alternative measures such as the pseudo-spectral radius or the field of values of the iteration matrix have been proposed in the literature. It is well known that, in general, the error, as measured in the L_∞ norm, may increase initially. As a result, in [5] [6], the initial error reduction was calculated by considering the matrix norm of the iteration matrix, $||M^{-1}N||_\infty$. It has already been demonstrated [4] that considerable benefits are derived by using iterative schemes which conform to the natural direction of the problem. In fact, when the diffusion coefficient is sufficiently small, a correctly conforming method will converge after only a few iterations.

In the case of the simple problem being considered, it is shown in [5] [6] that for mesh spacings h which satisfy $\epsilon \leq .8\hat{p}h$ the error reduction for the *backward* Gauss-Seidel (BGS) is given by

$$||M^{-1}N||_\infty \leq \frac{2\epsilon}{\check{p}h^2}\ , \tag{3}$$

and thus if $\epsilon < \check{p}h^2/2$ the error reduction is monotonic. Note that (3) implies very rapid convergence if $\epsilon \ll h$. The *forward* Gauss-Seidel (FGS), on the other hand does not exhibit this property. Conversely, if $p(x) \leq 0$, FGS exhibits rapid convergence under appropriate conditions and BGS does not.

Consistent with the philosophy of ϵ-uniform methods, in this paper, we are interested in characterizing the behaviour of iterative methods over a full range of ϵ, rather than only for small ϵ.
The precise behaviour of the error for each iteration, as a function of N and ϵ, can be represented in the following form:

Theorem 1: *Let z^h be the solution of the upwind difference scheme with n points approximating problem 1 with perturbation parameter ϵ. If z_i is the solution of the i^th iterate of the Gauss-Seidel process then the error is bounded as follows.*
For FGS the error is of the following form:

$$\begin{aligned}
|z^h - z_i^h| &\leq M, \quad 0 \leq i \leq M_0(\epsilon n^2 + n), && (4a)\\
|z^h - z_i^h| &\leq Mexp(-\frac{m_0[i - M_0(\epsilon n^2+n)]}{1+\epsilon^2 n^2}), \quad i \geq M_0(\epsilon n^2+n). && (4b)
\end{aligned}$$

In the case of BGS the error is of the following form:

$$\begin{aligned}
|z^h - z_i^h| &\leq M\epsilon n^2, \qquad 0 \leq i \leq m_1, \quad \epsilon n^2 \leq m_2, && (5a)\\
|z^h - z_i^h| &\leq M, \quad 0 \leq i \leq M_0 \epsilon n^2, \quad \epsilon n \leq m_5, && (5b)\\
|z^h - z_i^h| &\leq M[m_3\epsilon^{-2}n^{-2}]^{-m_4(i-M_0\epsilon n^2)}, \quad i \geq M_0\epsilon n^2, \quad \epsilon n \leq m_5 && (5c)\\
|z^h - z_i^h| &\leq M, \quad 0 \leq i \leq M_0\epsilon n^2, \quad \epsilon n > m_5, && (5d)\\
|z^h - z_i^h| &\leq Mexp(-m_3\epsilon^{-2}n^{-2}(i - M_0\epsilon n^2)), \quad i \geq M_0\epsilon n^2, \quad \epsilon n > m_5. && (5e)
\end{aligned}$$

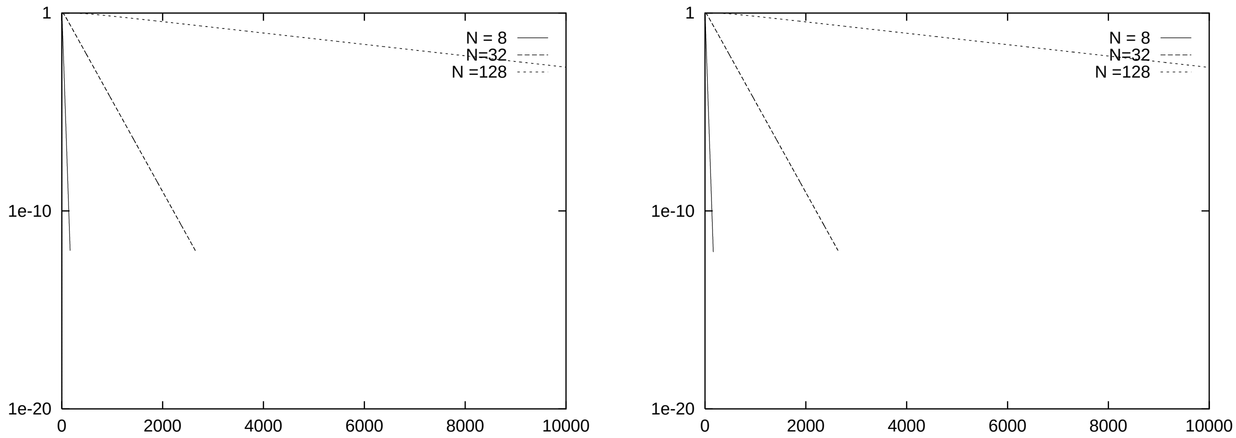
In these expression M is a generic constant independent of n and ϵ, and M_0, m_0, m_1, m_2 and m_3 are constants independent of n and ϵ.
This result can be proved using a maximum principle argument and a sufficiently complex barrier function technique similar to those in [7].

Numerical Results

We illustrates this behaviour for the model problem

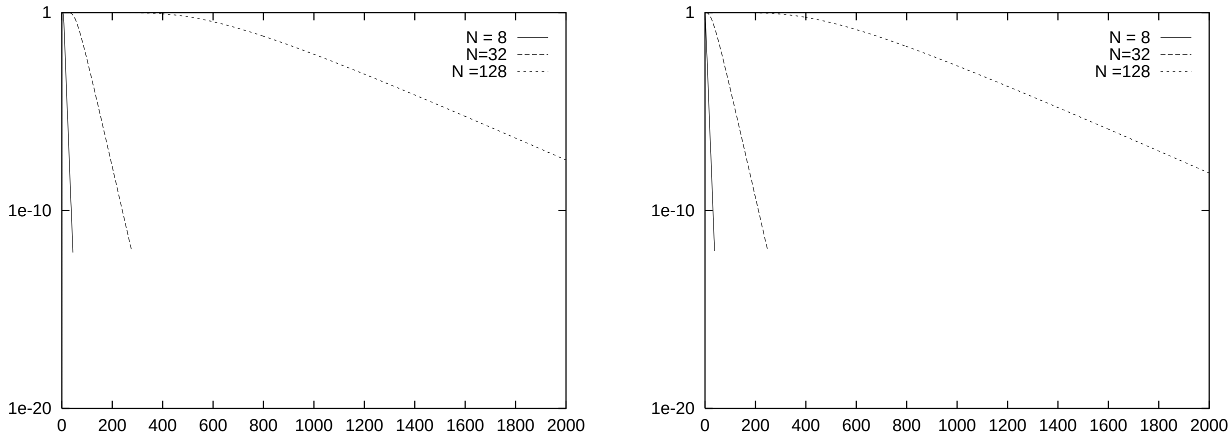
$$\epsilon u''(x) + u'(x) = 2, \;\; 0 < x < 1, \tag{6}$$

$$u(0) = -1, \quad u(1) = 1.$$



(a) FGS (b) BGS

Figure 1: Graph of the log of error against iterations for $\epsilon = 1/2$

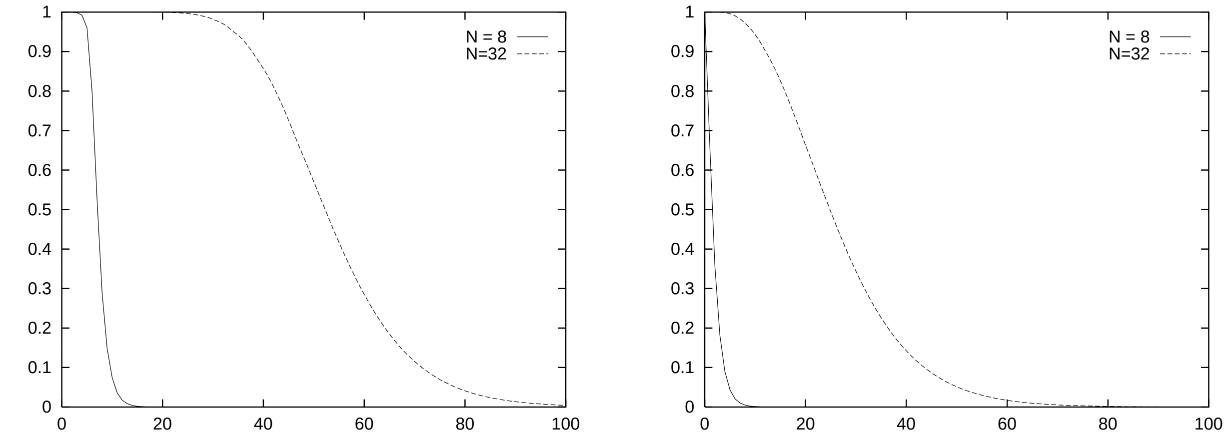


(a) FGS (b) BGS

Figure 2: Graph of the log of error against iterations for $\epsilon = 1/32$

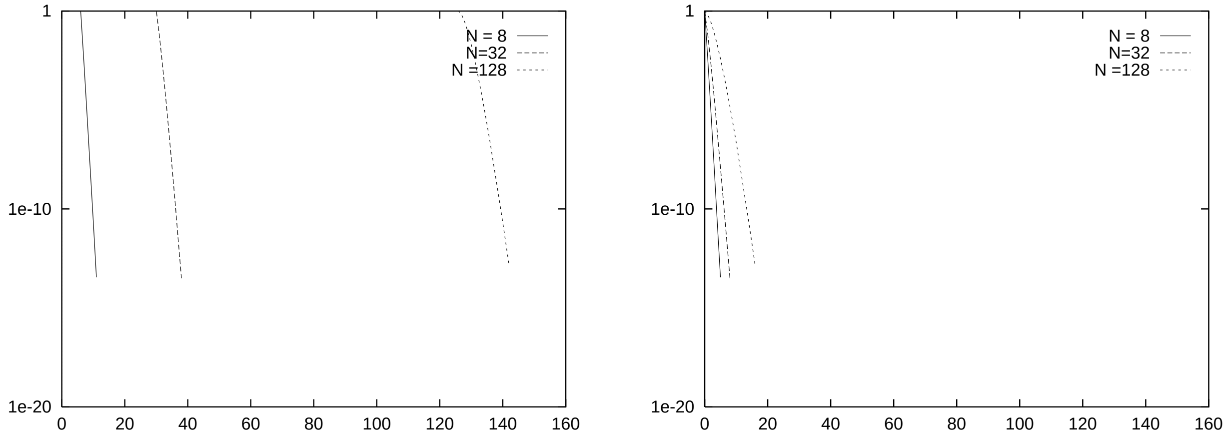
It should be noted that this problem has an exact solution $u(x) = 2x - 1$ and thus exhibits no boundary layers. For all the results exhibited, the initial guess is chosen so that the initial error at each point is 1.

First we illustrate the reduction of the error with the number of iterations for various values of h and ϵ. In each case the left hand graph in the figure gives the errors or logarithm of the errors of FGS, which is the normal direction used, and the right hand graph gives the equivalent result for BGS. Figure 1 gives the behaviour for $\epsilon = 1/2$, Figures 2, and 3 for $\epsilon = 1/32$ and Figure 4 for $\epsilon = 1/128^2$ respectively. Figure 1 illustrates that in the classical case, where ϵ is of order 1 (in this case .5) there is no significant difference between the performance of FGS and BGS. In this case the initial behaviour of FGS is given by (4a) and of BGS by (5d), while, after order n^2 iterations, the error begins to decay exponentially as given by (4b) and (5e) respectively. Figures 2 and 3 illustrate a transitional case, where



(a) FGS (b) BGS

Figure 3: Graph of the error against iterations for $\epsilon = 1/32$



(a) FGS (b) BGS

Figure 4: Graph of the log of error against iterations for $\epsilon = 1/128^2$

$\epsilon = 1/32$. For $n = 8$, the initial behaviour of FGS for approximately 10 iterations is given by (4a) and that of BGS for approximately 2 iterations by (5c), whereas the asymptotic rates are given by (4b) and (5d) respectively. For $n = 128$ on the other hand, the initial rate for FGS is given by (4a) for approximately 640 iterations, and that for BGS by (5d) for about 512 iterations. In the case of Figure 4, we have reached the strongly convection dominated flow case, since $\epsilon = 1/128^2 = .00006103515625$. Thus for $n = 8$ and $n = 32$, $\epsilon n \approx \epsilon n^2 \approx 0$, and even in the case of $n = 128$, $\epsilon n^2 << n$. Hence, in all cases the error for FGS is given by (4a) for approximately n iterations and then decays exponentially by (4b). In the case of BGS, the error is given initially by (5a) or (5b), but only for 0 or 1 iterations following which it is given by (5c). Thus it converges to a given accuracy in approximately n less iterations than FGS. As can be seen from these graphs, the asymptotic rates are similar as indicated by the ultimate slopes of the logarithm graphs.

Conclusions

We have presented a theoretical result on the convergence rates of forward and backward Gauss-Seidel for a model singular perturbation problem and illustrated this result with numerical examples. These result enable a qualitative prediction of the error and are useful in determining the relative performance of the two methods. For small ϵ, in particular for $\epsilon << n^2$, the backward Gauss-Seidel is significantly faster to converge to a given tolerance than the forward Gauss-Seidel. In general, the method which correctly follows the flow inherent in the problem will exhibit this property. These results are of interest not only if Gauss-Seidel is used as a iterative solver but, perhaps more importantly, if it is used as a smoother in a multigrid method.

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