A MULTIGRID ALGORITHM FOR NONLINEAR MONOTONE ELLIPTIC PROBLEMS

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1. Introduction. Let Ω be an open bounded set in $I\!R^2$ with a Lipschitz boundary $\partial\Omega$. Given $p \in (2, \infty)$, $f \in W^{-1,p}(\Omega)$ and the operator

$$A : W_0^{1,p}(\Omega) \longrightarrow W^{-1,p}(\Omega)$$

defined by

$$< Au, v > = \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v d\Omega, \quad \forall u, v \in W_0^{1,p}(\Omega), \tag{1.1}$$

where $W^{-1,p}(\Omega)$ is the dual of the Sobolev space $W_0^{1,p}(\Omega)$, and $|.|^2 = (.,.)_{I\!R^2}$, we introduce in this paper a multigrid algorithm to compute the numerical solution of the following problem:

$$(P) \qquad \begin{cases} \text{Find } u \in W_0^{1,p}(\Omega) \text{ such that for all } \quad v \in W_0^{1,p}(\Omega) \\ < Au, v >=< f, v > . \end{cases}$$

Problem (P) is the weak formulation of the Dirichlet problem for the p-Laplacian

$$\begin{cases} -\nabla.(|\nabla u|^{p-2}\nabla u) = f \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.2}$$

It is well established that A is uniformly monotone and continous, i.e., there exist positive constants c_1 and c_2 such that

$$< Au - Av, u - v > \geq c_1 ||u - v||_{1,p}^p \tag{1.3}$$

and

$$||Au - Av||_{-1,p} \leq c_2 ||u - v||_{1,p} (||u||_{1,p} + ||v||_{1,p})^{p-1} \tag{1.4}$$

where $||.||_{1,p}$ and $||.||_{-1,p}$ denote the norms of $W_0^{-1,p}(\Omega)$ and $W^{1,p}(\Omega)$ respectively. Further, A is also the subgradient of the strictly convex functional

$$\begin{aligned} J^* : W_0^{1,p}(\Omega) &\longrightarrow I\!R \\ J^*(u) &= \frac{1}{p} \int_\Omega |\nabla u|^p d\Omega. \end{aligned} \tag{1.5}$$

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So that problem (P) is equivalent to the minimization problem [2]

$$J(u) = \min_{v \in W_0^{1,p}(\Omega)} J(v), \tag{1.6}$$

where $J(v) = J^*(v) - \int_\Omega f v d\Omega$.

$J(u)$ is Gateaux differentiable and satisfy the following relations:

$$J'(u,v) = \int_\Omega |\nabla u|^{p-1} \nabla u . \nabla v - \int_\Omega f v d\Omega, \quad \forall u, v \in W_0^{1,p}(\Omega). \tag{1.7}$$

$$\begin{aligned} J''(u,v,w) = \int_\Omega |\nabla u|^{p-2} \nabla v . \nabla w d\Omega + \\ (p-2) \int_\Omega |\nabla u|^{p-4} (\nabla u . \nabla v)(\nabla u . \nabla w) d\Omega. \end{aligned} \tag{1.8}$$

The application of Holder inequality on (1.7) and (1.8) yields the following bounds.

$$|J'(u,v)| \leq c_3 ||u||_{1,p}^{p-1} ||v||_{1,p} + ||f||_{-1,p} ||v||_{1,p}, \tag{1.9}$$

$$|J''(u,v,w)| \leq c_4 ||u||_{1,p}^{p-2} ||v||_{1,p} ||w||_{1,p}. \tag{1.10}$$

Note that from relations (1.7)-(1.10) it follows that $J(u)$ is twice Gateaux differentiable and has a bounded Hessian $H(u)$. By virtue of (1.3) and (1.10) it is easy to establish that $H(u)$ is directional positive definite, i.e., for $u, v \in W_0^{1,p}(\Omega)$ there exists $\alpha \in (0,1)$ depending on u and v and positive constants K_1 and K_2 such that

$$K_1 ||v||_{1,p}^p \leq |H(u + \alpha v)| \leq K_2 ||u + \alpha v||_{1,p}^{p-2} ||v||_{1,p}^2 \tag{1.11}$$

The latter property is of interest to us for the application of the nonlinear conjugate gradient method in the pre-smoothing and post-smoothing steps of the multigrid algorithm proposed in this paper

Glowinski and Marrocco [3] developed an augmented Lagrangian algorithm to solve the finite element formulation of problem (P). Such an algorithm is robust, however, its rate of convergence is slow. More recently, Wei [4] based on the inequalities (1.11) satisfied by the Hessian $H(u)$, proposed a gradient algorithm to find the numerical solution to problem (P), whereas Barret and Liu [1] have used Polak-Ribiere non linear conjugate gradient method (PRNCG) and reported that this method was far superior to Wei ´ s gradient method.

We propose in this paper a FAS multigrid algorithm that, based on the properties of the functional $J(u)$, uses the PRNCG method with exact searching in the pre-smoothing and post-smoothing steps.

2. Finite Element Formulation. Let Ω_h be a polygonal domain that approximates Ω. Ω_h is divided into a finite number NE of small triangles T_j satisfying

i) $\bar{\Omega}_h = \bigcup_{j=1}^{NE} T_j$,

ii) for $i \neq j$,
$$T_i \bigcap T_j = \begin{cases} \emptyset, \text{ void or} \\ P_k, \text{ a common vertex, or} \\ \Gamma_{ij}, \text{ a common side.} \end{cases}$$

iii) Let $h_i = diam(T_i)$ and $\rho_i = supdiam(Sph \subset T_i)$, then there exists a constant $\sigma > 0$, such that
$$\frac{h_i}{\rho_i} < \sigma, \quad \forall i$$

iv) Let $\{P_k\}_{k=1}^{M}$, $M > 0$, be the set of vertices in $\bar{\Omega}_h$. If $P_i \in \partial\Omega_h$ then $P_i \in \partial\Omega$.

We define a finite dimensional space associated with the partition $D_h = \{T_i\}_{i=1}^{NE}$ as:
$$V = \{v_h \in C(\bar{\Omega}_h) : v_h|_{T_j} \in P_1(T_j), \forall T_j \in D_h\},$$
where $P_1(T)$ is the space of linear polynomials defined on T. V has the following approximation properties (Ciarlet). For $m = 0$ or 1 and for all $T_i \in D_h$, we have:

a) for $q \in [1, \infty]$, $s \in [1, \infty]$, provided $W^{2,s}(T_j) \hookrightarrow W^{m,q}(T_j)$ and $W_2^{2,1}(T_j) \hookrightarrow C(\bar{T}_j)$,
$$\inf_{\chi \in V} \|v - \chi\|_{m,q,T_i} \leq Ch^{2(\frac{1}{q} - \frac{1}{s})} h^{2-m} \|v\|_{2,s,T_j}, \quad \forall v \in W^{2,s}(T_j).$$

b) For $q > 2$
$$\inf_{\chi \in V} \|v - \chi\|_{m,q,T_i} \leq Ch^{1-m} \|v\|_{1,q,T_j}, \quad \forall v \in W^{1,q}(T_j).$$

Let $V^0 = W_0^{1,p}(\Omega) \bigcap V$, the finite element approximation of (1.2) is formulated as:

$$(P_h) \qquad \begin{cases} \text{Find } u_h \in V^0 \text{ such that for all } v_h \in V^0 \\ < Au_h, v_h >=< f, v_h > \end{cases}$$

The corresponding minimization problem is then:

$$\begin{array}{l} \text{Find } u_h \in V^0 \text{ such that} \\ J_h(u_h) = \min_{v_h \in V^0} J_h(v_h). \end{array}$$

where
$$J_h(v_h) = \frac{1}{p} \int_{\Omega_h} |\nabla v_h|^p d\Omega - \int_{\Omega_h} f v_h d\omega.$$

By virtue of the inclusion $V^0 \subset W_0^{1,p}(\Omega)$, the properties of the operator A and the functional $J(v)$ are inherited by the corresponding discrete A and $J_h(v_h)$, respectively.

3. The Multigrid Algorithm. We describe in this section the multigrid algorithm to solve (P). In order to simplify the notation, we shall drop hereafter the subscript h from the discrete variables unless otherwise stated.
Let $\{D_k\}_{k=o}^m$ be a sequence of partitions of Ω_h. Given D_0, for $1 \leq k \leq m$, D_k is obtained from D_{k-1} by a regular subdivision that joins the edge midpoints of any triangle by edges to form new triangles. So that, the discretization parameter h_k of D_k is $h_k = \frac{1}{2}h_{k-1}$. Let V_k^0 be the finite element space associated with D_k, the following inclusion relations hold: $D_{k-1} \subset D_k$ and $V_{k-1}^0 \subset V_k^0$, $\forall k$.
To define the multigrid algorithm we shall introduce auxiliary operators. For $k = 0, 1, \ldots m$, the mesh-dependent inner product $(.,.)_k$ on V_k^0 is defined by

$$(v, w)_k = h_k^2 \sum_{j=1}^{M_k} v(P_j)w(P_j)$$

where $\{P_j\}_{j=1}^{M_k}$ is the set of vertices of D_k. The operator $A_k : V_k^0 \longrightarrow V_k^0$ is defined by

$$(A_k w, \Phi)_k = < Aw, \Phi >, \quad \forall \Phi \in V_k^0$$

Also, the functional $J_k : V_k^0 \longrightarrow I\!R$ and the operator $I_k^{k-1} : V_k^0 \longrightarrow V_{k-1}^0$ are defined by

$$J_k(w) = \frac{1}{p}\int_{\Omega_h} |\nabla w|^p d\Omega - (f, w)_k, \quad \forall w \in V_k^)$$

and

$$(I_k^{k-1} w, v)_{k-1} = (w, v)_k, \quad \forall w \in V_k^0, \quad v \in V_{k-1}^0,$$

respectively.
The operator I_k^{k-1} is known as fine-to-coarse grid transfer operator. We also need the natural injection operator I_{k-1}^k from coarse-to-fine grid, which is defined as

$$I_{k-1}^k v = v \in V_{k-1}.$$

I_{k-1}^k is the transpose of I_k^{k-1} with respect to the inner product $(.,.)_{k-1}$.

The k-th level V-cycle iteration $MG(k, z_0, g_k)$ is the approximate to

$$A_k z = g_k,$$

with initial gess z_0.
For $k = 0$, $MG(0, z, g)$ is the solution obtained by application of the PRNCG algorithm to the minimization problem

$$J_0(u) \leq \min_{v \in V_0^0} J_0(v).$$

Thus, $MG(0, z_0, g) = \text{PRNCG}(J_0(v), v_0)$.
For $k > 0$, $MG(k, z_0, g)$ is obtained recursively in the following steps:

1) Pre-smoothing Step: For $1 \leq l \leq m_1$, let

$$z_l = \mathrm{PRNCG}(J_k(v_l), z_{l-1}).$$

2) Correction Step: Let

$$z_{m_1+1} = z_{m_1} + I_{k-1}^k(\bar{z} - I_k^{k-1}(z_{m_1})),$$

where

$$\bar{z} = MG(k-1, q, h)$$

and

$$q = I_k^{k-1} z_{m_1},$$

$$h = I_k^{k-1}(g - A_k z_{m_1}) + A_k q.$$

3) Post-Smoothing Step: For $m_1 + 2 \leq l \leq m_1 + m_2 + 1$, let

$$z_l = PRNCG(J_k(v), z_{m_1+1}).$$

Here m_1 and m_2 are positive integers.

The k-th level iteration is applied in the strategy of full multigrid algorithm invented by A. Brandt.

The Full Multigrid Algorithm

For $k = 0$, $u_0 = PRNCG(J_0(v), v_0)$.
For $k \geq 1$, the approximate solutions u_k are obtained recursively from

$$z = I_{k-1}^k(u_{k-1})$$
$$u_k = MG(k, z, f_k).$$

Some comments are in order. We use the $PRNCG$ as smoother for various reasons. The first one is due to the equivalence between the minimization of the functional $J_k(v)$ and the weak solution of problem (P_h). The second reason is that Newton method may fail as smoother because the Hessian $H(u)$ is in general semi-positive definite. Our third reason has to do with the difficulties in computing the discrete derivatives of the finite element formulation.
For convenience of the reader, we describe briefly the $PRNCG$ as is implemented in our calculations.

$\mathbf{PRNCG(J_k(v), v_0)}, \quad \mathbf{k = 0, 1, \ldots m}.$

Step 0. Choose $v_0 \in \mathbb{R}^{M_k}$. If $J_k'(v_0) = 0$, stop;
else GO TO Step 1.
Step 1. Set $j = 0,$, $g_0 = h_0 = -J_k'(z_0)$.

Step 2. Compute λ_j (Brent' s method) such that

$$J_k(z_j + \lambda_j h_j) = \min\{J_k(z_j + \lambda h_j) : \quad \lambda \geq 0\}.$$

Step 3. Set

$$z_{j+1} = z_j + \lambda_j h_j$$

Step 4. Compute

$$J_k'(z_{j+1}) = 0, \quad \text{Stop;} \quad \text{else,} \quad \text{Set}$$

$$g_{i+1} = -J_k'(z_{j+1}),$$
$$h_{j+1} = g_{j+1} + \gamma_j h_j$$

with

$$\gamma_j = \frac{(g_{j+1} - g_j, g_{j+1})}{|g_j|^2}.$$

Set $j = j + 1$ and GO TO Step 2.

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